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Brief paper

Distributed predictive control based on Gaussian process models☆

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ABSTRACT

A suboptimal approach to distributed NMPC is proposed based on Gaussian process models of the interconnected systems dynamics and taking into account the imposed constraints. The suggested method is based on a sequential linearization of the nonlinear system dynamics and finding a suboptimal solution of the resulting Quadratic Programming (QP) problem by using distributed iterations of the dual accelerated gradient method. The main advantages of the distributed approach are that it allows the computation of the suboptimal control inputs to be done autonomously by the subsystems without the need for centralized optimization and it has a simple software implementation. The proposed method is illustrated with simulations on the simplified model of a sewer system.

1. Introduction

Nonlinear Model Predictive Control (NMPC) has become the accepted methodology to solve complex control problems related to process industries (Grüne & Pannek, 2011; Magni et al., 2009; Mayne et al., 2000; Rawlings et al., 2017). It involves the solution at each sampling instant of a finite horizon optimal control problem subject to nonlinear system dynamics and state and input constraints. *Stochastic* MPC problems are formulated in the applications where the system to be controlled is described by a stochastic model. Thus, the approaches in van Hessem et al. (2001) and Yan and Bitmead (2005) are based on *linear* state space models with stochastic parameters and/or additive noise. In Kouvaritakis et al. (2010), an approach to stochastic MPC for *linear* systems is proposed that explicitly uses probabilistic distributions to guarantee recursive feasibility and stability of the closed-loop system. In Farina et al. (2015), a method to output-feedback MPC of stochastic *linear* systems affected by a possibly unbounded additive noise is suggested. In Lorenzen et al. (2017), a sampling-based stochastic MPC algorithm is proposed.

Further, several approaches to *stochastic nonlinear* MPC have been developed. Thus, in Bradford and Imsland (2019), a shrinking horizon NMPC algorithm has been proposed accounting for the stochastic uncertainties to optimize a probabilistic objective subject to chance constraints. The stochastic NMPC approach in Sehr and Bitmead (2017) uses the particle filter equations to estimate the required statistics. In Schlüter and Allgöwer (2020), an NMPC strategy for stochastic systems with state- and input-dependent, finite-support disturbances subject to individual chance constraints has been suggested. Very useful reviews on stochastic MPC approaches can be found in Mesbah (2016) and Heirung et al. (2018). It should be noted that the stochastic MPC approaches mentioned so far are based on *parametric* probabilistic models. Alternatively, the stochastic systems can be modelled with *nonparametric* models which can offer a significant advantage compared to the parametric models, which is related to their ability to provide information about prediction uncertainties. The Gaussian process model is an example of a nonparametric probabilistic black-box model and up to now it has been applied to model mainly static nonlinearities. The underlying approaches to the modelling of dynamic systems by using Gaussian processes (GP) can be found in Kocijan et al. (2005) and Solak et al. (2003). Several methods for *stochastic* NMPC based on GP model have been proposed (e.g. Bradford et al., 2020; Grancharova et al., 2008; Hewing et al., 2019; Likar & Kocijan, 2007; Murray-Smith et al., 2003). An overview of using GP in the modelling and control of dynamic systems is provided in Kocijan (2016).

The papers above consider *centralized stochastic* MPC. However, the centralized solution of MPC problems for large-scale systems may be impractical due to the topology of the plant and data communication, and the large number of decision variables. Several methods for *distributed/decentralized* MPC for *deterministic* linear/nonlinear systems have been developed (e.g. Alessio et al., 2011; Christofides et al., 2013; Giselsson et al., 2013;

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Grancharova & Johansen, 2014; Grancharova et al., 2016; Maestre & Negenborn, 2014). In Grancharova and Johansen (2014), an approach to *distributed* NMPC of *deterministic* nonlinear systems with separable coupled dynamics has been suggested, which is based on distributed on-line optimization. Another more general method for *distributed deterministic* NMPC has been developed in Grancharova et al. (2016), which does not assume separability of the couplings between the subsystems and finds distributedly a suboptimal solution of the resulting optimization problem. In Dai et al. (2018), a cooperative *distributed stochastic* MPC approach is proposed for multiple dynamically decoupled *linear* subsystems with both parameter uncertainty and stochastic disturbances and coupled probabilistic constraints. So far, no approach to *distributed* NMPC of dynamically coupled nonlinear systems described by Gaussian processes has been proposed.

The contribution of this paper is a suboptimal approach to *distributed* NMPC based on Gaussian process (GP) models of the interacting systems dynamics (referred to as distributed GP-NMPC approach). In difference to the methods in Grancharova and Johansen (2014), Grancharova et al. (2016), where the subsystems are described by *deterministic* state–space models, here *stochastic* interacting subsystems are considered whose dynamics are modelled with nonparametric (GP) models. A GP-NMPC problem for the overall system is formulated, which is first locally approximated by a *linear centralized* GP-MPC and then solved distributedly by applying an extension of the method in Grancharova et al. (2016).

The paper is structured as follows. In Section 2, the modelling of interconnected stochastic dynamic systems with Gaussian processes is described and a centralized GP-NMPC problem is formulated. In Section 3, an approach to distributed GP-NMPC is proposed. The developed approach is illustrated with simulations on a stochastic system representing simplified model of a sewer system in Section 4. The conclusions are gathered in Section 5.

2. Model predictive control problem for stochastic interconnected systems

2.1. Preliminaries on modelling with Gaussian processes

A Gaussian process is an example of the use of a flexible, probabilistic, nonparametric model which directly provides us with uncertainty predictions. Its use and properties for modelling are reviewed in Rasmussen and Williams (2006). A Gaussian process is a collection of random variables which have a joint multivariate Gaussian distribution. Assuming a relationship of the form $y = f(z)+\xi$ (with ξ being a stochastic noise) between an input $z \in \mathbb{R}^D$ and output $y \in \mathbb{R}$, we have $y(1), y(2), \ldots, y(L) \sim \mathcal{N}(0, \mathbf{\Sigma})$, where $\Sigma_{pq} = \mathrm{Cov}(y(p), y(q)) = C(z(p), z(q))$ gives the covariance between the output points $y(p)$ and $y(q)$ corresponding to the input points $z(p)$ and $z(q)$. Thus, the mean $\mu(z)$ (usually assumed to be zero) and the covariance function $C(z(p), z(q))$ fully specify the Gaussian process. Note that the covariance function $C(z(p), z(q))$ can be any function with the property that it generates a positive definite covariance matrix. Forms of covariance functions suitable for different applications can be found in Rasmussen and Williams (2006). For a given problem, the hyperparameters of the covariance function are identified using the data at hand.

Consider a set of L D-dimensional input vectors $\mathbf{Z} = [z(1), z(2), \ldots, z(L)]^{\mathrm{T}}$ and a vector of output data $Y = [y(1), y(2), \ldots, y(L)]^{\mathrm{T}}$. Based on the data $(\mathbf{Z}, Y)$, and given a new input vector z^*, we wish to estimate the probability distribution of the corresponding output y^*. Unlike other models, there is no model parameter determination as such, within a fixed model structure. With this model, most of the effort consists in *tuning* the hyperparameters of the covariance function. One of the approaches is by maximizing the log-likelihood of the parameters, i.e. optimally determine them from the evidence (or marginal distribution) of the GP posterior. This is called the empirical Bayes or Type II maximum likelihood optimization. The maximum a posteriori (MAP) estimate of the hyperparameters equals the maximum marginal likelihood estimate of the evidence (not parameters) of the GP posterior (for more details refer to Rasmussen & Williams, 2006 and Kocijan, 2016). Based on a training set $\mathbf{Z}$, a covariance matrix $\mathbf{K}$ of size $L \times L$ is determined. As already mentioned before, the aim is to estimate the probability distribution of the corresponding output y^* at some new input vector z^*. For a new test input z^*, the posterior distribution of the corresponding output is $y^*|z^*, (\mathbf{Z}, Y)$ and is Gaussian, with mean and variance (Rasmussen & Williams, 2006):

$$
\begin{aligned}
&E\{y^*\} = \mu(z^*) = c(z^*)^{\mathrm{T}}\mathbf{K}^{-1}Y \\
&\mathrm{var}\{y^*\} = \sigma^2(z^*) = c_0(z^*) - c(z^*)^{\mathrm{T}}\mathbf{K}^{-1}c(z^*) + v_0
\end{aligned} \tag{1}
$$

where $c(z^*) = [C(z(1), z^*), \ldots, C(z(L), z^*)]^{\mathrm{T}}$ is the vector of covariances between the test and training cases, $c_0(z^*) = C(z^*, z^*)$ is the covariance between the test input and itself.

Gaussian processes can model static nonlinearities and can therefore be used for modelling of dynamic systems if delayed input and output signals are used as regressors (Kocijan, 2016). In such cases an autoregressive (NARX) model is considered, where the current predicted output depends on previous estimated outputs and previous inputs:

$$
\begin{aligned}
&z(t) = [\hat{y}(t-1), \ldots, \hat{y}(t-l), u(t-1), \ldots, u(t-l)]^{\mathrm{T}} \\
&\hat{y}(t) = \tilde{f}(z(t)) + \eta(t)
\end{aligned} \tag{2}
$$

where t denotes consecutive number of data sample, l is a given lag, and $\eta(t)$ is the prediction error.

2.2. Gaussian processes based model of interconnected dynamic systems

Consider a stochastic system composed by the interconnection of M stochastic subsystems with overall state and overall control input:

$$
x(t) = [x_1(t), x_2(t), \ldots, x_M(t)] \in \mathbb{R}^n, \quad n = \sum_{i=1}^{M} n_i \tag{3}
$$

$$
u(t) = [u_1(t), u_2(t), \ldots, u_M(t)] \in \mathbb{R}^m, \quad m = \sum_{i=1}^{M} m_i \tag{4}
$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and the control input, related to the ith subsystem. It is assumed that the topology of the overall system is known, i.e. it is known which subsystems interact with each other through their inputs, states or both. Let the dynamics of the subsystems be described by *uncertain* nonlinear discrete-time models:

$$
x_i(t+1) = h_i(x(t), u(t)) + \xi_i(t), \quad i = 1, 2, \ldots, M \tag{5}
$$

where $h_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ is a nonlinear continuous function and $\xi_i(t) \in \mathbb{R}^{n_i}$ are Gaussian disturbances. From (5) it can be seen that the state evolution of each subsystem depends on the overall state and the overall input, i.e. the subsystems have coupled dynamics both through their states and inputs. With known topology of interactions, the uncertainty consists in that the analytical expressions of $h_i(x, u)$ are not known and neither are the mean values and the covariances of the disturbances $\xi_i(t)$, $i = 1, 2, \ldots, M$. The relationships (5) are represented in the form:

$$
y_i(t) = h_i(z(t)) + \xi_i(t), \quad i = 1, 2, \ldots, M \tag{6}
$$

where $y_i(t) = x_i(t+1) \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, M$ and $z(t) = [x(t), u(t)] \in \mathbb{R}^{n+m}$. Suppose that we have an output data set $Y_{i,j} = [y_{i,j}(0), y_{i,j}(1), \ldots, y_{i,j}(L-1)]$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, n_i$ corresponding to an input data set $\mathbf{Z} = [z(0), z(1), \ldots, z(L-1)]$. Assume that the relationship (6) is approximated with Gaussian processes with distributions:

$$Y_{i,1} \sim \mathcal{N}(0, \mathbf{\Sigma}_{i,1}), \; Y_{i,2} \sim \mathcal{N}(0, \mathbf{\Sigma}_{i,2}), \; \ldots,$$
$$Y_{i,n_i} \sim \mathcal{N}(0, \mathbf{\Sigma}_{i,n_i}), \; i = 1, 2, \ldots, M \tag{7}$$

where the covariance functions $\Sigma_{i,1,pq} = \mathrm{Cov}_{i,1}(y_{i,1}(p), y_{i,1}(q)) = C_{i,1}(z(p), z(q)), \ldots, \Sigma_{i,n_i,pq} = \mathrm{Cov}_{i,n_i}(y_{i,n_i}(p), y_{i,n_i}(q)) = C_{i,n_i}(z(p), z(q))$ with $p = 0, 1, \ldots, L-1$, $q = 0, 1, \ldots, L-1$ depend on the given input and output data sets. Having obtained the Gaussian process model (7), the probability distribution of the output $y_i(L) = [y_{i,1}(L), \ldots, y_{i,n_i}(L)]$ corresponding to a new input $z(L)$ can be determined as described in the previous section:

$$\begin{aligned} &y_{i,1}(L)|z(L), (\mathbf{Z}, Y_{i,1}) \sim \mathcal{N}(\mu(y_{i,1}(L)), \sigma^2(y_{i,1}(L))) \\ &\vdots \\ &y_{i,n_i}(L)|z(L), (\mathbf{Z}, Y_{i,n_i}) \sim \mathcal{N}(\mu(y_{i,n_i}(L)), \sigma^2(y_{i,n_i}(L))) \\ &i = 1, 2, \ldots, M \end{aligned} \tag{8}$$

In (8), $\mu(y_{i,j}(L))$ and $\sigma^2(y_{i,j}(L))$ denote respectively the mean and the variance of the output variable $y_{i,j}(L)$, $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, n_i$. We introduce the vectors $\mu_{y_i}(L) = [\mu(y_{i,1}(L)), \ldots, \mu(y_{i,n_i}(L))]$, $\sigma^2_{y_i}(L) = [\sigma^2(y_{i,1}(L)), \ldots, \sigma^2(y_{i,n_i}(L))]$ and the matrix $\mathbf{Y}_i = [Y_{i,1}, Y_{i,2}, \ldots, Y_{i,n_i}]$. Then, the relation (6) is represented:

$$y_i(L)|z(L), (\mathbf{Z}, \mathbf{Y}_i) \sim \mathcal{N}(\mu_{y_i}(L), \sigma^2_{y_i}(L))$$
$$i = 1, 2, \ldots, M \tag{9}$$

Based on (9), a multi-step ahead prediction can be obtained:

$$y_i(L+k)|z(L+k), (\mathbf{Z}, \mathbf{Y}_i) \sim \mathcal{N}(\mu_{y_i}(L+k), \sigma^2_{y_i}(L+k))$$
$$i = 1, 2, \ldots, M, \; k = 0, 1, \ldots, N-1 \tag{10}$$

where N is a prediction horizon. Suppose the initial overall state $x(t) = x_{t|t}$ and control inputs $u(t+k) = u_{t+k}$, $k = 0, 1, \ldots, N-1$ are given. Then, by taking into account that $y_i(t) = x_i(t+1)$ and $z(t) = [x(t), u(t)]$, from (10) we obtain the probability distribution of the predicted states $x_{i,t+k+1|t}$, $i = 1, 2, \ldots, M$, $k = 0, 1, \ldots, N-1$ which correspond to the given initial state $x_{t|t}$ and control inputs u_{t+k}, $k = 0, 1, \ldots, N-1$:

$$x_{i,t+k+1|t}|x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{i,t+k+1|t}), \sigma^2(x_{i,t+k+1|t}))$$
$$i = 1, 2, \ldots, M, \; k = 0, 1, \ldots, N-1 \tag{11}$$

The 95% confidence interval of the random variable $x_{i,t+k+1|t}$ is $[\mu(x_{i,t+k+1|t}) - 2\sigma(x_{i,t+k+1|t}); \mu(x_{i,t+k+1|t}) + 2\sigma(x_{i,t+k+1|t})]$, where $\sigma(x_{i,t+k+1|t})$ is the standard deviation. Concerning the predictions with the GP model (11) the following should be mentioned. Simulation of NARX models is obtained iteratively where the prediction from the previous step is used in the inputs for the next step considered continuing indefinitely or until the end of the horizon of interest. This makes the inputs uncertain and we, therefore, have to result to approximations (Kocijan, 2016). Currently the simulation procedures in GP modelling practice are roughly divided to: (1) naive simulation, (2) approximation of statistical moments, (3) numerical approximation. Naive simulation underestimates the uncertainty, since only the mean is propagated to future steps. Since naive simulation is computationally least demanding, it is used for the proof of concept that is presented in this paper. The size of uncertainty, even though not equivalent to other approximation, still indicates the potential trust in model predictions.

For convenience, the predictions of the mean values of $x_{i,t+k+1|t}$ are represented as:

$$\mu(x_{i,t+k+1|t}) = E\{f_{\mathrm{GP},i}(x_{t+k|t}, u_{t+k})\}$$
$$i = 1, 2, \ldots, M, \; k = 0, 1, \ldots, N-1 \tag{12}$$

where the function $f_{\mathrm{GP},i}(x_{t+k|t}, u_{t+k})$ is defined by the Gaussian process (GP) model (1) and $E\{\cdot\}$ denotes mathematical expectation. The representation (12) is used further in Section 3.

The constraints imposed on the subsystems are:

$$x_i(t) \in \mathcal{X}_i, \; u_i(t) \in \mathcal{U}_i, \; i = 1, 2, \ldots, M \tag{13}$$

where $\mathcal{X}_i$ and $\mathcal{U}_i$ are the admissible sets, and the following assumption is made:

A1. The admissible sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are bounded polyhedral sets, i.e. they are defined by:

$$\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \le d_i^x\}$$
$$\mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \le d_i^u\} \tag{14}$$

where $C_i^x \in \mathbb{R}^{n_{c,x_i} \times n_i}$, $C_i^u \in \mathbb{R}^{n_{c,u_i} \times m_i}$, $d_i^x \in \mathbb{R}^{n_{c,x_i}}$, $d_i^u \in \mathbb{R}^{n_{c,u_i}}$ and n_{c,x_i} and n_{c,u_i} are the number of constraints imposed on x_i and u_i, respectively.

It can be seen from (14) that the constraints imposed on the subsystems are not coupled. In order to consider coupled constraints, the approach described in the next section should be slightly modified.

2.3. Formulation of centralized GP-NMPC problem

It is supposed that a full measurement $\bar{x} = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_M]$ of the overall state is available at the current time t. The optimal regulation problem is considered where the goal is to steer the overall state of the system (5) to the set-point $x_{\mathrm{sp}} = [x_{1,\mathrm{sp}}, x_{2,\mathrm{sp}}, \ldots, x_{M,\mathrm{sp}}]$. For the current overall state $\bar{x}$, the regulation GP-NMPC solves the optimization problem:

Problem P1 (Centralized GP-NMPC):

$$V^{\mathrm{opt}}(\bar{x}) = \min_U J(U, \bar{x}) \tag{15}$$

subject to $x_{t|t} = \bar{x}$ and:

$$\mu(x_{i,t+k|t}) - 2\sigma(x_{i,t+k|t}) \in \mathcal{X}_i$$
$$i = 1, \ldots, M, \; k = 1, \ldots, N \tag{16}$$
$$\mu(x_{i,t+k|t}) + 2\sigma(x_{i,t+k|t}) \in \mathcal{X}_i$$
$$i = 1, \ldots, M, \; k = 1, \ldots, N \tag{17}$$
$$u_{i,t+k} \in \mathcal{U}_i, \; i = 1, \ldots, M, \; k = 0, 1, \ldots, N-1 \tag{18}$$
$$x_{i,t+k+1|t}|x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{i,t+k+1|t}), \sigma^2(x_{i,t+k+1|t}))$$
$$i = 1, \ldots, M, \; k = 0, 1, \ldots, N-1 \tag{19}$$
$$x_{t+k|t} = [x_{1,t+k|t}, \ldots, x_{M,t+k|t}], \; k = 0, 1, \ldots, N \tag{20}$$
$$u_{t+k} = [u_{1,t+k}, \ldots, u_{M,t+k}], \; k = 0, 1, \ldots, N-1 \tag{21}$$

with $U = [u_t, u_{t+1}, \ldots, u_{t+N-1}]$ and the global cost function:

$$\begin{aligned} J(U, \bar{x}) &= \sum_{i=1}^{M} J_i(U, \bar{x}) \\ &= \sum_{i=1}^{M} \sum_{k=0}^{N} l_i(\mu(x_{i,t+k|t}), u_{i,t+k}) \end{aligned} \tag{22}$$

Here, $J_i(U, \bar{x})$ is the local cost function for the ith subsystem with the stage cost defined by:

$$\begin{aligned} l_i(\mu(x_{i,t+k|t}), u_{i,t+k}) &= \|\mu(x_{i,t+k|t}) - x_{i,\mathrm{sp}}\|^2_{Q_i} \\ &+ \|u_{i,t+k} - u_{i,\mathrm{sp}}\|^2_{R_i} \end{aligned} \tag{23}$$

where Q_i, $R_i \succ 0$ are symmetric weighting matrices, $u_{i,\mathrm{sp}}$ is the steady state value of the control input of the subsystem corresponding to $x_{i,\mathrm{sp}}$, and N is a finite horizon.

3. Distributed GP-NMPC by sequential linearization

3.1. Approximation of the GP-NMPC problem by linear MPC problem

Most often, the covariance function in the stochastic model (1) is chosen to be the Gaussian function. Therefore, the Gaussian processes (GP) based prediction models (12) of the subsystems are nonlinear in $x_{t+k|t}$ and u_{t+k}. Here, these models are locally approximated by linear models. First, consider the deviations of the state and the control input vectors from their set-point values:

$$\tilde{x}_{i,t+k} = x_{i,t+k} - x_{i,\mathrm{sp}}\,,\ \tilde{u}_{i,t+k} = u_{i,t+k} - u_{i,\mathrm{sp}}$$
$$k = 0,\ 1,\ \ldots\ ,\ N-1\,,\ i = 1,\ \ldots\ ,\ M \tag{24}$$

The models (12) are represented as:

$$\mu(\tilde{x}_{i,t+k+1|t}) = -x_{i,\mathrm{sp}} + E\{f_{\mathrm{GP},i}(\tilde{x}_{t+k|t} + x_{\mathrm{sp}}, \tilde{u}_{t+k} + u_{\mathrm{sp}})\} \tag{25}$$
$$k = 0,\ 1,\ \ldots\ ,\ N-1\,,\ i = 1,\ \ldots\ ,\ M$$

where $\tilde{u}_{t+k} = [\tilde{u}_{1,t+k},\ \ldots\ ,\ \tilde{u}_{M,t+k}]$, $\tilde{x}_{t+k} = [\tilde{x}_{1,t+k},\ \ldots\ ,\ \tilde{x}_{M,t+k}]$, $u_{\mathrm{sp}} = [u_{1,\mathrm{sp}},\ \ldots\ ,\ u_{M,\mathrm{sp}}]$ and $x_{\mathrm{sp}} = [x_{1,\mathrm{sp}},\ \ldots\ , x_{M,\mathrm{sp}}]$. Let at time t, $\tilde{U}_i^0 = [\tilde{u}_{i,t}^0, \tilde{u}_{i,t+1}^0,\ \ldots\ ,\ \tilde{u}_{i,t+N-1}^0]$ and $\tilde{X}_i^0 = [\tilde{x}_{i,t|t}^0, \tilde{x}_{i,t+1|t}^0,\ \ldots\ , \tilde{x}_{i,t+N-1|t}^0]$ be given trajectories of the deviated control input and the mean of deviated state of the ith subsystem for the prediction horizon N. Taylor series expansion of the right-hand side of the model (25) about the point $(\tilde{U}_i^0, \tilde{X}_i^0)$ leads to the locally linear model:

$$\mu(\tilde{x}_{i,t+k+1}) = E\{\sum_{j=1}^{M}(A_{ij,t+k}\tilde{x}_{j,t+k} + B_{ij,t+k}\tilde{u}_{j,t+k}) + g_{i,t+k}\} \tag{26}$$
$$k = 0,\ 1,\ \ldots\ ,\ N-1\,,\ i = 1,\ \ldots\ ,\ M$$

where the matrices $A_{ij,t+k}$, $B_{ij,t+k}$ and the vector $g_{i,t+k}$ are computed as:

$$A_{ij,t+k} = \nabla_{\tilde{x}_j} f_{\mathrm{GP},i}(\tilde{x}_{t+k|t}^0 + x_{\mathrm{sp}}, \tilde{u}_{t+k}^0 + u_{\mathrm{sp}})$$
$$B_{ij,t+k} = \nabla_{\tilde{u}_j} f_{\mathrm{GP},i}(\tilde{x}_{t+k|t}^0 + x_{\mathrm{sp}}, \tilde{u}_{t+k}^0 + u_{\mathrm{sp}})$$
$$g_{i,t+k} = -x_{i,\mathrm{sp}} - \sum_{j=1}^{M}(A_{ij,t+k}\tilde{x}_{j,t+k}^0 + B_{ij,t+k}\tilde{u}_{j,t+k}^0) + f_{\mathrm{GP},i}(\tilde{x}_{t+k}^0 + x_{\mathrm{sp}}, \tilde{u}_{t+k}^0 + u_{\mathrm{sp}}) \tag{27}$$
$$k = 0,\ 1,\ \ldots\ ,\ N-1\,,\ i, j = 1,\ \ldots\ ,\ M$$

In (27), $\tilde{u}_{t+k}^0 = [\tilde{u}_{1,t+k}^0,\ \ldots\ ,\ \tilde{u}_{M,t+k}^0]$ and $\tilde{x}_{t+k|t}^0 = [\tilde{x}_{1,t+k|t}^0,\ \ldots,\ \tilde{x}_{M,t+k|t}^0]$. It can be observed that (26)–(27) is a linear time-varying approximation of the model (25).

The following assumption is made:

A2. The standard deviations $\sigma(x_{i,t+k+1|t})$ predicted with the GP model (11) satisfy:

$$\sigma(x_{i,t+k+1|t}) \le \sigma_{\max,i}$$
$$i = 1,\ \ldots\ ,\ M\,,\ k = 0,\ \ldots\ ,\ N-1 \tag{28}$$

where $\sigma_{\max,i}\,,\ i = 1,\ \ldots\ ,\ M$ are known.

It should be noted that $\sigma_{\max,i}\,,\ i = 1,\ \ldots\ ,\ M$ is a control-design parameter provided by a control-system designer based on simulations with the GP models obtained on the entire range of data used for the model identification. It can be interpreted as the upper bound (hard constraint) on admissible trust in model predictions.

The following tightened constraint sets are introduced:

$$(1-\delta)\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \le (1-\delta)d_i^x\} \tag{29}$$

$$(1-\delta)\mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \le (1-\delta)d_i^u\} \tag{30}$$

where $\delta \in (0,\ 1)$ is the amount of relative constraint tightening. The reason for the tightening is related to the fact that a suboptimal solution of the GP-NMPC problem will be found (see Section 3.3) and it should be ensured that it will keep the original constraints. Then, the tightened constraint sets of the state and the control input deviations from their set-point values are:

$$\tilde{\mathcal{X}}_i = \{\tilde{x}_i \in \mathbb{R}^{n_i} \mid C_i^x \tilde{x}_i \le (1-\delta)d_i^x - C_i^x x_{i,\mathrm{sp}}\} \tag{31}$$
$$\tilde{\mathcal{U}}_i = \{\tilde{u}_i \in \mathbb{R}^{m_i} \mid C_i^u \tilde{u}_i \le (1-\delta)d_i^u - C_i^u u_{i,\mathrm{sp}}\} \tag{32}$$

For the locally linear dynamics (26)–(27) with initial state $\tilde{x}^0 = [\tilde{x}_{1,t|t}^0,\ \ldots\ ,\ \tilde{x}_{M,t|t}^0]$, the linear MPC problem is formulated:

Problem P2 (Centralized linear MPC):

$$V^*(\tilde{x}^0) = \min_{\tilde{U}} J(\tilde{U}, \tilde{x}^0) \tag{33}$$

subject to $\tilde{x}_{t|t} = \tilde{x}^0$ and:

$$\mu(\tilde{x}_{i,t+k|t}) - 2\sigma_{\max,i} \in \tilde{\mathcal{X}}_i$$
$$i = 1,\ \ldots\ ,\ M\,,\ k = 1,\ \ldots\ ,\ N \tag{34}$$
$$\mu(\tilde{x}_{i,t+k|t}) + 2\sigma_{\max,i} \in \tilde{\mathcal{X}}_i$$
$$i = 1,\ \ldots\ ,\ M\,,\ k = 1,\ \ldots\ ,\ N \tag{35}$$
$$\tilde{u}_{i,t+k} \in \tilde{\mathcal{U}}_i\,,\ i = 1,\ \ldots\ ,\ M\,,\ k = 0, 1,\ \ldots\ ,\ N-1 \tag{36}$$
$$\mu(\tilde{x}_{i,t+k+1}) = E\{\sum_{j=1}^{M}(A_{ij,t+k}\tilde{x}_{j,t+k} + B_{ij,t+k}\tilde{u}_{j,t+k}) + g_{i,t+k}\}\,,\ k = 0,\ 1,\ \ldots\ ,\ N-1\,,\ i = 1,\ \ldots\ ,\ M \tag{37}$$

with $\tilde{U} = [\tilde{u}_t,\ \tilde{u}_{t+1},\ \ldots\ ,\ \tilde{u}_{t+N-1}]$ and the cost function given by:

$$J(\tilde{U}, \tilde{x}^0) = \sum_{i=1}^{M}\sum_{k=0}^{N}[\|\mu(\tilde{x}_{i,t+k|t})\|_{Q_i}^2 + \|\tilde{u}_{i,t+k}\|_{R_i}^2] \tag{38}$$

It should be noted that the state constraints (34)–(35) are expectation constraints, since they are imposed on the predicted mean of the state variables. Constraining the control operation to the region of 95% uncertainty using design parameter $\sigma_{\max,i}$ is based on two ideas: 1) we do not want the controller to go out of the region where the model is not good enough, 2) we have a model good enough within acceptable $\sigma_{\max,i}$ set by the control designer. Good enough is qualitative judgement of the designer, based on quantitative measures of the model quality. Therefore, the control design is conditioned on the process's model quality. When solving problem P2, the constraints (28) are taken into account as in Kocijan (2016).

3.2. Representation and solution of the linear MPC problem as a distributed quadratic programming problem

Similar to Grancharova et al. (2016), by stacking all decision variables (the control input trajectory and the state trajectory along the horizon) into one vector $S \in \mathbb{R}^{n_S}$ with dimension $n_S = \sum_{i=1}^{M} N(n_i + m_i)$:

$$S = [\mu(\tilde{x}_{1,t+1|t}), \tilde{u}_{1,t},\ \ldots\ ,\ \mu(\tilde{x}_{1,t+N|t}), \tilde{u}_{1,t+N-1},\ \vdots\ \mu(\tilde{x}_{M,t+1|t}), \tilde{u}_{M,t},\ \ldots\ ,\ \mu(\tilde{x}_{M,t+N|t}), \tilde{u}_{M,t+N-1}] \tag{39}$$

the optimization problem P2 can be written as a QP problem:

Problem P3 (QP problem):

$$V^*(\tilde{x}^0) = \min_{S} \frac{1}{2} S^{\mathrm{T}} \bar{H} S \tag{40}$$

subject to:

$$\bar{A}S = \bar{B}\tilde{x}^0 - \bar{G} \tag{41}$$

$$\bar{C}S \leq \bar{d} \tag{42}$$

Here, $\bar{H}$, $\bar{A}$, $\bar{B}$, $\bar{G}$, $\bar{C}$, and $\bar{d}$ are matrices/vectors with appropriate dimensions, which are easily obtained from the matrices/vectors involved in the formulation of problem P2.

The linear MPC problem P2 can be solved distributedly by applying the dual accelerated gradient algorithm in Giselsson et al. (2013). The distribution is enabled by solving the dual problem to problem P3, which is created by introducing dual variables $\lambda \in \mathbb{R}^{n_{\bar{A}}}$ for the equality constraints (41) and dual variables $\eta \in \mathbb{R}^{n_{\bar{C}}}$ for the inequality constraints (42). It is shown in Giselsson et al. (2013) that the dual problem is:

$$\max_{\lambda,\, \eta \geq 0} D(\tilde{x}^0, \lambda, \eta) \tag{43}$$

where $D(\tilde{x}^0, \lambda, \eta)$ is the dual cost function:

$$D(\tilde{x}^0, \lambda, \eta) = -\frac{1}{2}(\bar{A}^{\mathrm{T}}\lambda + \bar{C}^{\mathrm{T}}\eta)^{\mathrm{T}}\bar{H}^{-1}(\bar{A}^{\mathrm{T}}\lambda + \bar{C}^{\mathrm{T}}\eta) - \lambda^{\mathrm{T}}(\bar{B}\tilde{x}^0 - \bar{G}) - \eta^{\mathrm{T}}\bar{d} \tag{44}$$

In order to perform distributedly the iterations of the dual gradient method, the vector $S_i \in \mathbb{R}^{n_{S_i}}$ ($n_{S_i} = N(n_i + m_i)$) of decision variables, associated to the ith subsystem, is introduced:

$$S_i = [\mu(\tilde{x}_{i,t+1|t}), \tilde{u}_{i,t}, \ \dots \ , \mu(\tilde{x}_{i,t+N|t}), \tilde{u}_{i,t+N-1}] \tag{45}$$

Let λ_i and η_i be the dual variables for the equality and the inequality constraints, related to the ith subsystem. The distributed iterations of the dual gradient method are:

$$S_i^r = -\bar{H}_i^{-1}\Big(\sum_{j=1}^{M} \bar{A}_j^{\mathrm{T}}\lambda_j^r + \bar{C}_i^{\mathrm{T}}\eta_i^r\Big) \tag{46}$$

$$\bar{S}_i^r = S_i^r + \frac{r-1}{r+2}(S_i^r - S_i^{r-1}) \tag{47}$$

$$\lambda_i^{r+1} = \lambda_i^r + \frac{r-1}{r+2}(\lambda_i^r - \lambda_i^{r-1}) + \frac{1}{L}(\bar{A}_i\bar{S}^r - (\bar{B}_i\tilde{x}_i^0 - \bar{G}_i)) \tag{48}$$

$$\eta_i^{r+1} = \max\Big(0, \eta_i^r + \frac{r-1}{r+2}(\eta_i^r - \eta_i^{r-1}) + \frac{1}{L}(\bar{C}_i\bar{S}_i^r - \bar{d}_i)\Big) \tag{49}$$

$$i = 1, \ 2, \ \dots \ , M$$

where $\bar{H}_i$, $\bar{A}_i$, $\bar{B}_i$, $\bar{G}_i$, $\bar{C}_i$, $\bar{A}_j^i$ and $\bar{d}_i$ are related to the ith subsystem and represent submatrices/subvectors of $\bar{H}$, $\bar{A}$, $\bar{B}$, $\bar{G}$, $\bar{C}$, $\bar{d}$ in problem P3, r is the iteration number and $L = \|[\bar{A}^{\mathrm{T}}, \bar{C}^{\mathrm{T}}]^{\mathrm{T}}\bar{H}^{-1}[\bar{A}^{\mathrm{T}}, \bar{C}^{\mathrm{T}}]\|$ is the Lipschitz constant to the gradient of the dual function (44).

3.3. Algorithm for distributed GP-NMPC by sequential linearization and distributed quadratic programming

Here, an algorithm for distributed GP-NMPC is suggested which is based on achieving Nash equilibrium between the interconnected subsystems. It represents a modification of the distributed deterministic NMPC algorithm in Grancharova et al. (2016) and includes two loops. In the outer loop, the GP models (12) of the stochastic nonlinear subsystems are locally approximated with linear models (26)–(27) about the current guess for the control input trajectory and the corresponding trajectory of state mean. Then, in the inner loop, a suboptimal solution to the resulting QP problem P3 is found by applying the distributed iterations (46)–(49) of the dual accelerated gradient method. The iterations continue until the overall system reaches Nash equilibrium.

Before describing the algorithm, the following notation is introduced. Let $U(t) = [u_t, u_{t+1}, \ \dots \ , u_{t+N-1}]$ be the current update of the control input trajectory. Denote with $\mu(X(t)) = [\mu(x_{t+1|t}), \mu(x_{t+2|t}), \ \dots \ , \mu(x_{t+N|t})]$ the corresponding state mean trajectory of the overall system where the constituent $\mu(x_{i,t+1|t}), \ \dots \ , \mu(x_{i,t+N|t}), i = 1, \ \dots \ , M$ are predicted with the GP model (12) for initial state $x_{t|t} = \bar{x} = [\bar{x}_1, \ \dots \ , \bar{x}_M]$.

Then, the current update $S(t)$ of the decision variables can be easily constructed according to (24) and (39). Respectively, if updates S^r are obtained by performing the iterations (46)–(49), the corresponding update U^r of the control input trajectory can be extracted from it. Assume that a tolerance $\varepsilon > 0$ of achieving Nash equilibrium between the subsystems is specified, i.e. the iterations in the outer loop will terminate if the following condition is satisfied:

$$|J_i(U_2, \bar{x}) - J_i(U_1, \bar{x})| \leq \varepsilon, \ \forall i = 1, \ \dots \ , M \tag{50}$$

Here, U_1, U_2 and $J_i(U_1, \bar{x})$, $J_i(U_2, \bar{x})$, $i = 1, \ \dots \ , M$ are the control input trajectories and the local cost function values for the subsystems obtained in two sequential iterations in the outer loop of the algorithm.

Suppose that the design parameters, which are the tolerance ε, the number R of iterations (46)–(49) and the relative constraint tightening δ are specified. Then, the algorithm in Table 1 is used.

In general, it would be necessary to perform an offline study of the performance of the algorithm with different values of the parameters ε, R and δ in order to ensure that the computed suboptimal NMPC in closed-loop with the stochastic system (5) described by the GP model (12) will lead to feasibility, stability and desired performance.

4. Example

4.1. System description

Sewer networks are distributed systems that consist of pipes, sewer stretches (in-line storage), retention reservoirs (off-line storage) with overflow capabilities, and nodes for merging of flows from different catchments. Combined sewer networks collect domestic and industrial sewage as well as rainwater drainage. During rainfall, they may be overloaded, therefore, real-time control over the distribution of collected water in the sewer network is required (Marinaki & Papageorgiou, 2005). Optimal operation implies that at rain events, the whole retention capacity of all reservoirs in the sewer is used before overflows of polluted water or surface flooding take place somewhere in the network. On the other hand, during dry weather conditions, the sewer storage capacities can be used for the smoothing of peak discharges towards the wastewater treatment plant. To address the distributed system control problem, a simple yet realistic model was designed based on a simplified routing of model flows through a series of reservoirs (Schütze & Beck, 2002). In our case, the model with two interacting tanks was used (Fig. 1) to consider also the backwater effects that appear in the pipe when the amount of water in the system is too high to process downstream.

The mathematical description of the two-tank set-up in Fig. 1 is the following. We presume nonlinear valve characteristic:

$$S_1\frac{dh_1}{d\tau} = -k_1\sqrt{h_1(\tau) - h_2(\tau)} + \phi_1(\tau) \tag{51}$$

$$S_2\frac{dh_2}{d\tau} = k_1\sqrt{h_1(\tau) - h_2(\tau)} - \phi_2(\tau) \tag{52}$$

Here, τ is the continuous time, h_1 and h_2 are the heights of liquid in the tanks, S_1 and S_2 are the cross sectional areas of the tanks, ϕ_1 and ϕ_2 are the volumetric flows, and k_1 is valve characteristic

Table 1
Algorithm for distributed GP-NMPC.

1. Given ε, R and δ. Let $t = 0$, $U(t) = [u_{\text{sp}},\ u_{\text{sp}},\ \ldots,\ u_{\text{sp}}]$.
2. Let the state at time t be $\bar{x} = [\bar{x}_1,\ \ldots,\ \bar{x}_M]$.
3. Compute the state mean trajectory $\mu(X(t))$ corresponding to initial state $\bar{x}$ and control input trajectory $U(t)$ by using the GP model (12) and the associated local cost function values $J_{\text{new},i} := J_i(U, \bar{x}),\ i = 1,\ \ldots,\ M$ in (22). Form the vector $S(t)$ of decision variables.
4. **Do**
5. $J_{\text{old},i} := J_{\text{new},i},\ i = 1,\ \ldots,\ M$
6. Obtain a linearized model (26)–(27) of the model (12) around the trajectories $(U(t),\ \mu(X(t)))$.
7. **For** $r = 0,\ 1,\ \ldots,\ R$ **do**
8. **If** $r = 0$ **then**
9. Initialize iterations (46)–(49) with $S^{-1} = S(t)$, $\lambda^0 = \lambda^{-1} = 0$, $\eta^0 = \eta^{-1} = 0$.
10. **else**
11. Let $S^{r-1} := S^r$, $\lambda^{r-1} := \lambda^r$, $\lambda^r := \lambda^{r+1}$, $\eta^{r-1} := \eta^r$, $\eta^r := \eta^{r+1}$.
12. **end**
13. Run (46)–(49) distributedly by communicating S_i^r and λ_i^r, $i = 1,\ \ldots,\ M$ between interconnected subsystems and obtain S^r, λ^{r+1}, η^{r+1} for the overall system. Extract U^r from S^r.
14. **end**
15. Let $U(t) = U^R$.
16. Compute the state mean trajectory $\mu(X(t))$ corresponding to initial state $\bar{x}$ and control input trajectory $U(t)$ by using the GP model (12) and the associated local cost function values $J_{\text{new},i} := J_i(U, \bar{x}),\ i = 1,\ \ldots,\ M$ in (22). Form the vector $S(t)$ of decision variables.
17. **while** Nash equilibrium is reached $(|J_{\text{new},i} - J_{\text{old},i}| < \varepsilon,\ \forall i = 1,\ \ldots,\ M)$
18. Apply to the overall system the input $u(t) = [I\ 0\ \ldots\ 0]U(t)$.
19. Let $t = t + 1$ and go to step 2.

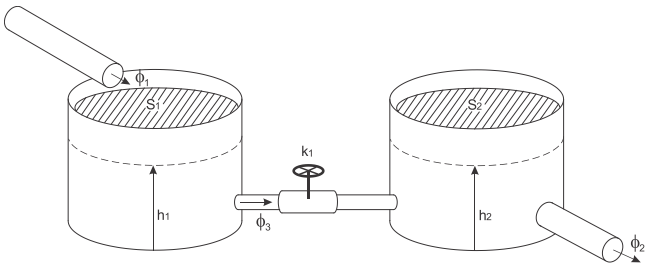

Fig. 1. The concept of interacting reservoirs for simplified modelling of relatively flat sewers where backwater phenomena may occur.

that relates to valve opening. The model (51)–(52) is discretized by applying the Euler method with sampling time T_s:

$$h_1(t+1) = h_1(t) - (T_s/S_1)k_1\sqrt{h_1(t) - h_2(t)} + (T_s/S_1)\phi_1(t) \tag{53}$$

$$h_2(t+1) = h_2(t) + (T_s/S_2)k_1\sqrt{h_1(t) - h_2(t)} - (T_s/S_2)\phi_2(t) \tag{54}$$

where t is the discrete time. For simulation purpose, the following parameter values are used: $T_s = 1$ min, $S_1 = 2.3130$ m^2, $S_2 = 2.1048$ m^2. For the simplified sewer system, $\phi_2(t)$ and $k_1(t)$ are the control inputs, $h_1(t)$ and $h_2(t)$ are the states, and $(T_s/S_1)\phi_1(t)$ is considered as a stochastic disturbance. The overall system consists of two interacting subsystems described by:

$$x_1(t+1) = x_1(t) - 0.4323u_1(t)\sqrt{x_1(t) - x_2(t)} + v(t) \tag{55}$$

$$x_2(t+1) = x_2(t) + 0.4751u_1(t)\sqrt{x_1(t) - x_2(t)} - 0.4751u_2(t) \tag{56}$$

Here, v is Gaussian disturbance, defined by $v = \mathcal{N}(0,\ 0.02^2)$. The two subsystems are coupled through their states and the control input u_1. The control goal is to adjust u_1 and u_2 so that the desired level of sewage in both tanks is maintained with minimal deviation from the set-point despite of input flow variations. Each of subsystems has been modelled as a GP model. In both cases the following model parameters squared-exponential covariance function is used:

$$C(z(t_p), z(t_q)) = v_1 \exp\left[-\frac{1}{2}\sum_{i=1}^{D} w_i(z_i(t_p) - z_i(t_q))^2\right] \tag{57}$$

where $z(t_p)$ and $z(t_q)$ are the input vectors to the GP model taken at the discrete times t_p and t_q. For subsystem (55), the input vector is $z(t_p) = [x_1(t_p),\ x_2(t_p),\ u_1(t_p)]$, while for subsystem (56) we have $z(t_p) = [x_1(t_p),\ x_2(t_p),\ u_1(t_p),\ u_2(t_p)]$. Inputs for models' identification are generated as random values with uniform distribution for all regressors. The estimated hyperparameters Θ^1 and Θ^2 of the GP models of subsystems (55) and (56) based on 1000 training samples are:

$$\begin{aligned}\Theta^1 &= [w_1^1,\ w_2^1,\ w_3^1,\ v_1^1] \\ &= [0.4275,\ 0.1989,\ 0.0005,\ 1.0167] \\ \Theta^2 &= [w_1^2,\ w_2^2,\ w_3^2,\ w_4^2,\ v_1^2] \\ &= [6.7230,\ 6.8256,\ 0.0008,\ 0.0046,\ 0.1707]\end{aligned} \tag{58}$$

Other parameters used for GP models are zero mean function and the exact inference method with Gaussian likelihood. The design parameters from Assumption A2 associated to the two subsystems are:

$$\sigma_{\max,1} = 0.025,\ \ \sigma_{\max,2} = 0.009 \tag{59}$$

and they are never violated with the suggested approach.

The set-point values of x_1, x_2 and the corresponding steady-state values of u_1, u_2 (obtained by optimization of the system behaviour for decaying disturbances) are:

$$\begin{aligned}&x_{1,\text{sp}} = 1.5 \text{ m},\ \ x_{2,\text{sp}} = 1.2 \text{ m} \\ &u_{1,\text{sp}} = 0.15 \text{ m}^2\sqrt{\text{m}}/\text{min},\ \ u_{2,\text{sp}} = 0.05 \text{ m}^3/\text{min}\end{aligned} \tag{60}$$

The constraints imposed on the system (55)–(56) are:

$$0.1 \le u_1(t) \le 0.2 \text{ m}^2\sqrt{\text{m}}/\text{min} \tag{61}$$

$$0 \le u_2(t) \le 0.1 \text{ m}^3/\text{min} \tag{62}$$

$$x_1(t) \le 1.8 \text{ m} \tag{63}$$

4.2. Simulation results

The prediction horizon in the centralized GP-NMPC problem is $N = 5$ and the weighting matrices in the local cost functions in (22) are $Q_1 = Q_2 = 10$, $R_1 = R_2 = 0.1$. The performance of the distributed GP-NMPC in closed-loop with the system (55)–(56) is studied for the case of a persistent stochastic disturbance v (related to the input flow ϕ_1). This pattern represents the usual time-variable release of domestic and industrial sewage to the sewer. The described algorithm is used to generate the

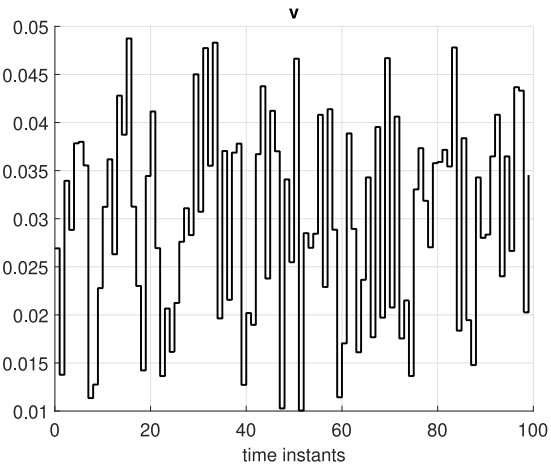

Fig. 2. The persistent stochastic disturbance v.

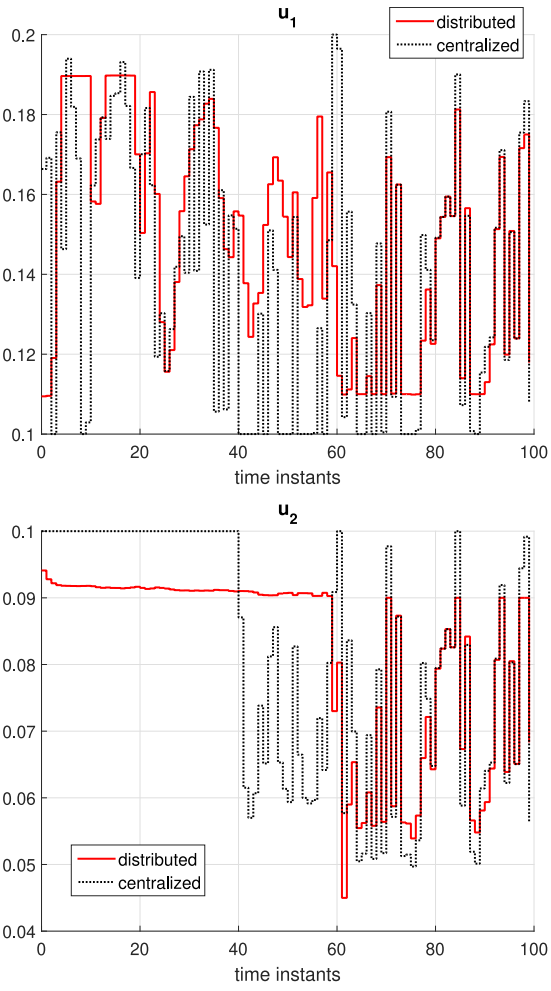

Fig. 3. Trajectories of the control inputs u_1 and u_2.

two control inputs for initial states of the subsystems $x_1(0) = 1.73$ m, $x_2(0) = 1.6$ m. The trajectories of the control inputs and the states obtained for the disturbance (shown in Fig. 2) are depicted in Fig. 3 and Fig. 4. The trajectories obtained with

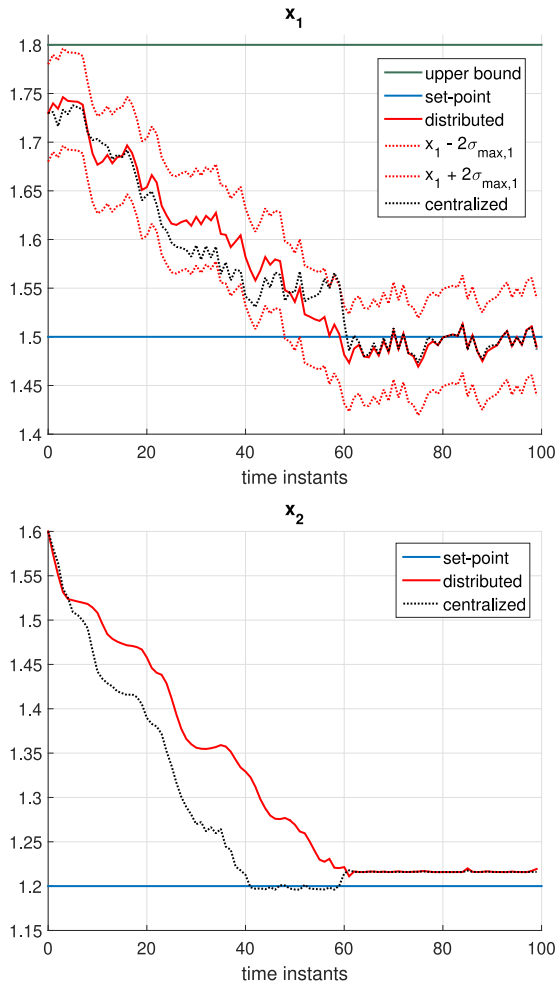

Fig. 4. Trajectories of the state variables x_1 and x_2.

the distributed GP-NMPC approach are compared to those corresponding to the centralized approach, which solves problem P1 at each time instant. The values of the design parameters in the algorithm are $\delta = 0.2$, $\varepsilon = 0.03$ and $R = 70$. It can be seen that the distributed GP-NMPC approach leads to feasible trajectories and the level of suboptimality is acceptable in sense that the suboptimality of the approach does not lead to a significant increase in the accumulated cost function value. In Fig. 4, the interval defined by $\pm 2\sigma_{\max,1}$ around the state x_1 of the system (55)–(56) in response to the distributed GP-NMPC is also depicted. It can be seen that this interval approaches the upper bound on x_1, but it does not violate it. The computational cost per sampling time of the distributed and the centralized approaches is compared and the results are presented in Table 2. The computations are performed on a 3.10 GHz AMD Ryzen 3 1200 quad-core processor. The distributed GP-NMPC has orders of magnitude less computational costs in comparison to the centralized approach. It also allows the computation of the suboptimal control inputs to be done autonomously by the subsystems without the need for centralized optimization and it has a simple software implementation. As a limitation, one may consider the necessity to perform an offline study of the performance of the proposed algorithm with different values of the design parameters in order to ensure closed-loop stability and desired performance.

Table 2
Comparison of GP-NMPC approaches.

Method	Average CPU time, s	Maximal CPU time, s
Distributed NMPC	0.10	0.34
Centralized NMPC	11.54	55.94

5. Conclusions

A suboptimal approach to distributed GP-NMPC is proposed based on Gaussian process models of the interconnected systems dynamics. It has a reduced complexity of the on-line computations and its simple software implementation makes it attractive for the implementation as embedded control. The simulations on the model of a sewer system demonstrate that the distributed GP-NMPC approach leads to feasible trajectories and the level of suboptimality is acceptable. Its applications to more complex systems and its further development are envisioned in the future.

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